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LETTER TO THE EDITOR

Long range percolation in one dimension

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Abstract. For lattice sites at integer values along the line let the probability of a bond between sites m and n be $p/|m-n|^s$, $m \neq n$, $0 \leqslant p < 1$, $1 < s \leqslant 2$. We prove that for $p < p_c^{(B)} = 1/2\zeta(s)$ there is no infinite cluster and that for $s > 2$ there is never an infinite cluster.

Long range percolation in one dimension has been considered by Zhang *et al* (1983). The sites are the integers $(0, \pm 1, \ldots)$ and for two sites m and n the probability that there exists a bond between them is $p/|m-n|^s$ with p and s parameters and $0 \leqslant p \leqslant 1$. If this problem should turn out to be similar to that of long range interactions for the one-dimensional Ising model (Kac 1978), then the interesting range of s would be 1 to 2 and one could hope to have a one dimensional but nevertheless non-trivial model for the study of percolation. Define p_c to be the critical value for the formation o infinite clusters. In this paper we show that there is no infinite cluster for $p < p_c^{(B)} = 1/2\zeta(s)$ where $p_c^{(B)}$ is the critical value on an appropriate Bethe lattice model that we define below and $\zeta(s)$ is the Riemann zeta function. Thus $p_c \geqslant p_c^{(B)} > 0$ for $s > 1$. Then if there is a regime in which infinite clusters do exist this problem will indeed have a non-trivial transition. Our result differs from that of Zhang *et al* (1983) who claim that $p_c = 0$ for $1 < s < 2$, i.e. that there is an infinite cluster for any non-zero p. We also prove that for $s > 2$ and $p < 1$ there is no infinite cluster. In passing we mention that preliminary Monte Carlo results suggest that the transition is fairly close to the Bethe lattice bound for s not close to 2.

The Bethe lattice model that we use for comparison is defined as follows. Instead of the usual finite number of branches emanating from each site we have a countable infinity. For each of these branches there is probability a_n ($n = 0, \pm 1, \pm 2, \ldots, 0 \leqslant a_n \leqslant 1$) that it contains a bond connecting the sites at its ends. It is not difficult to show (see appendix) that there is an infinite percolating cluster on this lattice if and only if $\Sigma a_n > 1$ so that the critical value is $\Sigma a_n = 1$. For comparison to the long range percolation model we shall take $a_n = p/|n|^s$, $n \neq 0$, $a_0 = 0$. The critical value of p is therefore $p_c^{(B)} = 1/2\zeta(s)$ where $\zeta(s) = \Sigma_{n=1}^{\infty} 1/n^s$, the Riemann zeta function. (This also turns out to be the critical value for a mean field theory for the one-dimensional model.)

In order to use the Bethe lattice for comparison we imagine that clusters are built on the long range model in the following way. We seek the cluster connected to the origin and to this end check the entire lattice for bonds between site 0 and site k, $k = \pm 1, \pm 2, \ldots$. Suppose there is success at sites $k_1, k_2, \ldots, k_N$. For each of these sites we check all possible bonds to other lattice sites but do not check whether they are connected to each other since this has no effect on the ultimate cluster constituents. Again for all the new sites we check their possible connections, not testing each other or sites incorporated at previous stages. The origin is then part of a finite cluster if and only if this process terminates.

The foregoing procedure, although describing long range percolation on a line, is quite similar to the way one would check for an infinite cluster on the Bethe lattice. The only difference is that for each new site incorporated on the Bethe lattice a full collection of branches is tried. For the long range model many possible branches are eliminated, and in fact it is those with the highest probability of success that are preferentially eliminated since however slowly $1/n^s$ drops off it does drop off. The bound is obtained by noting that if p is sufficiently small that the iterative procedure for the Bethe lattice *always* (in probability) terminates (i.e. $p<p_c^{(B)}=1/2\zeta(s)$) then for the long range model it *surely* always terminates.

To prove our contention that for $s>2$, $p<1$, there is no infinite cluster we consider the question of how many bonds cross some particular point on the line. Specifically, define A_{ij} to be a random variable taking the value 1 if there is a bond connecting i to j, zero otherwise. (Thus $\langle A_{ij}\rangle = p/|i-j|^s$, $i\neq j$.) Let $T(k)$ be the number of bonds crossing the point $k+\frac{1}{2}$, that is

$$T(k)=\sum_{\substack{i>k\\ j\leq k}} A_{ij}.$$

A value of k for which $T(k)=0$ would interrupt any putative infinite cluster. Moreover, a finite density of such points would prevent the existence of infinite clusters, however exotic (cf Newman and Schulman 1981). However, it is easy to see that for any k the probability that $T(k)=0$ is non-zero for $s>2$ and $p<1$. Thus

$$\mathrm{Prob}(T(k)=0)=\mathrm{Prob}\left(\prod_{\substack{i>k\\ j\leq k}}(1-A_{ij})=1\right)=\left\langle\prod_{\substack{i>k\\ j\leq k}}(1-A_{ij})\right\rangle=\prod_{\substack{i>k\\ j\leq k}}\left(1-\frac{p}{|1-j|^s}\right)$$

the last equality following from the independence of the A_{ij}'s. For $p<1$ no individual factor in the product vanishes and the infinite product vanishes only if the sum

$$\sum_{\substack{i>k\\ j\leq k}}|i-j|^{-s}$$

diverges, which it does not for $s>2$. It follows by translation invariance and ergodicity that there is a finite density of interruptions along the line. (Ergodicity can be established by replacing the pair labelled random variables by an equivalent set labelled by individual sites.)

Appendix. Bethe lattice with an infinity of branches and varying probabilities

The following demonstration is being provided because we do not know of any source in the literature. Arguments that are similar in spirit are given by Essam (1972, 1980). Let x_0 be a site on the Bethe lattice described above and let B_0 be the random variable taking the value one if there is an infinite cluster lying downstream from x_0 (or in Essam's (1972) terminology, there is an open walk of infinite length starting from x_0), zero otherwise. Then $\langle B_0\rangle=\rho$, the probability for the event just described. Let the branches emanating from x_0 be labelled by n and the sites they reach be labelled x_n. Let A_n be the random variable taking the value one in case the branch from x_0 to x_n has a bond, zero otherwise; let $\langle A_n \rangle = a_n$. Finally let B_n take the value one if x_n has an infinite cluster lying downstream of it, zero otherwise. The following relation is exact:

$$1 - B_0 = \prod_n (1 - A_n B_n). \tag{1}$$

All random variables on the right are independent of one another since each B_n depends only on variables downstream of x_n which are mutually disjoint and A_n is upstream of x_n. Moreover, the A_n are independent of one another. Therefore, taking expectation values we have

$$1 - \rho = \prod_n (1 - a_n \langle B_n \rangle). \tag{2}$$

All the $\langle B_n \rangle$ equal ρ since from any given site the infinite progression downstream looks exactly the same. Therefore ρ satisfies

$$1 - \rho = \prod_n (1 - \rho a_n). \tag{3}$$

The infinite product converges if Σa_n does, which we assume to be the case. $\rho = 0$ is always a solution and we seek the percolation threshold by looking for the condition that brings a non-trivial solution of (3) to zero. Expand (3):

$$1 - \rho = 1 - \rho \sum a_n + \rho^2 \sum_n a_n \sum_{m>n} a_m + O(\rho^3). \tag{4}$$

For $\rho \neq 0$ and to lowest order in ρ this yields

$$\rho = [(\sum a_n) - 1]/[\text{finite positive number}].$$

The denominator is finite since it is the double sum in (4) which is surely finite if Σa_n is. This establishes $\Sigma a_n = 1$ as the percolation threshold.

References

Essam J W 1972 in *Phase Transitions and Critical Phenomena* ed C Domb and M S Green (New York: Academic)

—— 1980 *Rep. Prog. Phys.* **43** 833

Kac M 1968 in *Mathematical Methods of Phase Transitions, Brandeis University Summer Institute* ed M Chretien, E P Gross and S Deser (New York: Gordon and Breach)

Newman C M and Schulman L S 1981 *J. Phys. A: Math. Gen.* **14** 1735

Zhang Z Q, Pu F C and Li B Z 1983 *J. Phys. A: Math. Gen.* **16** L85–90